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LETTER TO THE EDITOR

Symmetries and first integrals for dissipative systems

F Schwarz† and W-H Steeb‡

† GMD, Institut F1, Postfach 1240, 5205 St. Augustin, West Germany
‡ Universität Paderborn, Theoretische Physik, 4790 Paderborn, West Germany

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Abstract. The connection between symmetries, time independent and time dependent first integrals for autonomous systems of first-order ordinary differential equations is discussed. Several new results which have been obtained by applying a REDUCE package are reported.

Let $x = (x_1, \dots x_n)$ be n functions depending on the variable t and let

$$\Omega_\nu(t, x) \equiv \dot{x}_\nu - \omega_\nu(x) = 0 \qquad (\nu = 1, \dots, n) \tag{1}$$

be a system of n autonomous differential equations of first order. Throughout it is assumed that the ω_ν are polynomials in x. The infinitesimal generator U of a point symmetry is defined by

$$U = \xi(t, x)\, \partial/\partial t + \eta_\alpha(t, x)\, \partial/\partial x_\alpha. \tag{2}$$

Summation from 1 to n over twice occurring indices is always understood. The infinitesimals ξ and η_α have to be determined from the invariance condition $U^{(1)}\Omega_\nu = 0$ on $\Omega_\nu = 0$ where $U^{(1)}$ is the first extension of U. The following system of linear partial differential equations for ξ and the η_α's is obtained

$$\partial\eta_\nu/\partial t + \omega_\alpha\, \partial\eta_\nu/\partial x_\alpha - \eta_\alpha\, \partial\omega_\nu/\partial x_\alpha - \omega_\nu(\partial\xi/\partial t + \omega_\alpha\, \partial\xi/\partial x_\alpha) = 0. \tag{3}$$

The general solution of the system (3) determines the full symmetry group for equations (1). This is in most cases impossible.

Symmetries often make it possible to diminish the number of dependent variables. To this end the variables x have to be transformed such that the corresponding infinitesimal generator of the symmetry becomes a translation in one of the dependent variables. From this it is clear that only those symmetries are useful for the integration of (1) for which this transformation may actually be performed.

Let us now discuss several special solutions of (3) from this point of view:

(i) For arbitrary ω_α there exist the obvious solutions

$$\xi = \text{constant}, \qquad \eta_\nu = 0, \qquad t \text{ translations} \tag{4a}$$

$$\xi = f(t, x), \qquad \eta_\nu = f(t, x)\omega_\nu, \qquad f \text{ arbitrary}. \tag{4b}$$

These symmetries are useless for the integration of (1) for the reason mentioned above.

(ii) If $\eta_\nu = 0$ for all ν and ξ does not depend on t, then the system (3) reduces to

$$\omega_\alpha \, \partial\xi(\boldsymbol{x})/\partial x_\alpha = 0. \tag{5}$$

If $F(\boldsymbol{x})$ is a solution of (5) it is a time independent first integral and the system (1) allows the symmetry generator

$$U = F(\boldsymbol{x}) \, \partial/\partial t. \tag{6}$$

Symmetries of this kind are the most useful ones for the integration.

(iii) Again let $\eta_\nu = 0$ for all ν and an exponential time dependence for ξ be assumed, i.e.

$$\xi(t, \boldsymbol{x}) = \mathrm{e}^{kt} \xi_0(\boldsymbol{x}). \tag{7}$$

The system (3) reduces to

$$k\xi_0(\boldsymbol{x}) + \omega_\alpha \, \partial\xi_0(\boldsymbol{x})/\partial x_\alpha = 0. \tag{8}$$

The solutions of (7) and (8) are the time dependent first integrals which have been proposed by one of the authors (Steeb 1982). If such an integral $F(t, \boldsymbol{x})$ with exponential time dependency can be found, the system (1) again allows a symmetry generator of the form (6) with a time dependent function F.

More general symmetry generators may be obtained if no restrictions on ξ are imposed. This discussion shows the relation between time independent first integrals, time dependent first integrals with exponential time dependence and a general symmetry generator.

We now consider several autonomous systems of the form (1) and determine a certain class of their symmetries. It turns out that all calculations in this field are very lengthy. For this reason there have been developed two computer algebra packages (Schwarz 1984a, b). If polynomial dependency of a predefined degree in $\boldsymbol{x}$ is assumed, they determine the time independent and the time dependent first integrals and more general infinitesimal generators of a symmetry almost completely automatically. Most results described below have been obtained by applying these REDUCE packages.

We begin with the Lotka–Volterra equations for three competing populations which has been considered by Steeb and Erig (1983) and Steeb *et al* (1983), namely

$$\dot{x}_1 = x_1(1 + ax_2 + bx_3), \qquad \dot{x}_2 = x_2(1 - ax_1 + \overset{c}{\cancel{b}}x_3), \qquad \dot{x}_3 = x_3(1 - bx_1 - cx_2) \tag{9}$$ 5 3v

where a, b and c are real parameters. For arbitrary a, b, c there is the first integral

$$I(t, \boldsymbol{x}) = (x_1 + x_2 + x_3)\, \mathrm{e}^{-3t} \quad (-t) \tag{10}$$

In addition there is the integral

$$I(t, \boldsymbol{x}) = x_1^k x_2^l x_3^m \, \mathrm{e}^{-(k+l+m)t} \tag{11}$$

$\sum_{k,l,m} \Phi_{k,l,m}$ if k, l and m are determined from $ak - cm = 0$, $bk + cl = 0$. For special values of the parameters there exist time dependent integrals of higher order in $\boldsymbol{x}$. If $b = -a$, $c = a$ there is the third-order integral

(k+l+m=const)

$$I(t, \boldsymbol{x}) = [x_1^3 + x_2^3 + x_3^3 + 3x_1(x_2^2 + x_3^2) + 3x_2(x_1^2 + x_3^2) + 3x_3(x_1^2 + x_2^2)]\, \mathrm{e}^{-3t}. \tag{12}$$

If $b = -a$, $c = 2a$ there is the fourth-order integral

VAUD $\forall a, b, c$

$$I(t, \boldsymbol{x}) = [x_1^4 + x_2^4 + x_3^4 + 4x_1(x_2^3 + x_3^3) + 4x_2(x_1^3 + x_3^3) + 4x_3(x_1^3 + x_2^3) + 6(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 12x_1x_2x_3(x_2 + x_3)]\, \mathrm{e}^{-4t}. \tag{13}$$

VAUD $\forall a, b, c$